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Mixed-state entanglement of a three-level system with an intensity-dependent interaction

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Abstract
In this communication, the entanglement degree due to quasi-mutual entropy of a three-level atom interacting with a single cavity field is investigated. We consider the situation for which the three-level system is initially in a mixed state, whereas the field may start from either a coherent or a squeezed state. We present a derivation of the unitary evolution operator on the basis of the dressed-state formalism taking into account an arbitrary form of nonlinearity of the intensity-dependent coupling, by means of which we identify and numerically demonstrate the region of parameters where significantly large entanglement can be obtained. Most interestingly, it is shown that features of the degree of entanglement are influenced significantly by different forms of the nonlinearity. The atom and radiation subsystems exhibit alternating sets of collapses and revivals due to the initially mixed states of the atom and radiation employed here.

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1. Overview

Cavity quantum electrodynamics in the strong coupling regimes has become an important ground for testing fundamental thoughts of quantum physics. For recent advances in this field, for example, see [1] and references therein. Fortunately, owing to technological progress, experiments considered unrealistic up to recently, can now be carried out. As a result, various intriguing genuinely quantum effects traceable back to the superposition principle, entanglement, quantum interference, etc, are now within experimental reach. Experiments of increasing difficulty in cavity quantum electrodynamics over the last years have, for example, made it possible to test fundamental radiation–matter interaction models involving single atoms. Moreover, cavity field states possessing remarkable nonclassical features have been generated and detected. Such a stimulating situation essentially stems from two decisive advancements. The first is the invention of reliable protocols for the manipulation of single atoms. The second is the ability to produce desired bosonic environments on demand. This progress has led to the possibility of controlling the form of the coupling between individual atoms and an arbitrary number of bosonic modes. As a consequence, fundamental matter–radiation interaction models such as, for instance, the JC model and most of its numerous nonlinear multiphoton generalizations, have been realized or simulated in the laboratory and their dynamical features have been tested more or less in detail.

On the other hand, the concept of entanglement is probably the most striking feature of quantum mechanics. By definition a pure quantum state of two or more subsystems is said to be entangled if it is not a product of states of each component [2]. It is worth noting that while entanglement may be created only if there exists a direct or indirect interaction mechanism between the parts in play, generally speaking, an entangled state may describe a physical situation wherein two or more entangled single subsystems are decoupled. The behaviour of the system in such a condition is dominated by the appearance of quantum correlations which become rather puzzling and counterintuitive when referred to well-separated parts of the system. Entanglement is the underlying mechanism for the measurement of quantum observables [3] and is responsible for the occurrence of decoherence effects in the dynamics of quantum systems [4]. A method using quantum mutual entropy to measure the degree of entanglement in the time development of the Jaynes–Cummings model has been adopted in [5, 6]. Alioui *et al* [7] further provided a comparison of some criteria for mixed-state entanglement in the Jaynes–Cummings model. It has been shown in [8] that entanglement can always arise in the interaction of an arbitrarily large system in any mixed state with a single qubit in a pure state. They have demonstrated this feature using the Jaynes–Cummings interaction of a two-level atom in a pure state with a field in a thermal state.

The scope of this paper is essentially to examine the entanglement for an initial mixed state of the atom. A systematic study of the entanglement properties of the atom–field interaction that have emerged from the quantum relative entropy has not been performed in multi-level atoms. It is the objective of the present contribution, which is a progress report in character, to contribute to this systematic study. We shall here focus on what is perhaps the simplest situation in this context, namely a three-level atom interacting with a single cavity field, including an arbitrary form of the intensity-dependent coupling. Thus the present work sheds some light on the entanglement behaviour of multi-level systems when initially an entangled mixed state of the coupled system is considered and how this is affected by different parameters of the multi-level system. The scheme we are going to discuss exploits the passage of a single atom only through the cavity. We wish to underline from the beginning the relevance of this aspect from an experimental point of view. Preparing and controlling a single atom is certainly much easier to achieve with respect to the case when the manipulation of many atoms is required. In addition, taking into consideration the low efficiency [9] of the atomic state detectors today used in laboratory, conditional measurement procedures involving one atom only instead of many ones, have to be preferred. The dynamics of several Hamiltonian models describing such systems is exactly treatable and, in most cases, testable in the laboratory. Also, a more intriguing reason is that investigating these systems is likely to shed light on basic questions of quantum mechanics. The point to be appreciated is indeed that, studying such systems, one has the opportunity to induce entanglement and to control its evolution in a multipartite physical system. The physical scenario relative to the problems we shall be faced with in a quantum electrodynamics context, involves three-level atoms interacting, one at a time, with a single quantized electromagnetic mode sustained by a high-Q resonator. In this case, the dynamical properties of the system are investigated using a Hamiltonian model characterized

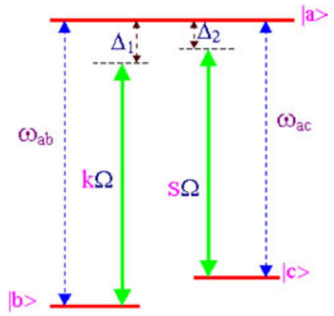

Figure 1. Schematic diagram of the degenerate Λ-type three-level atom interacting with a single-mode field. The levels $|a\rangle_A$, $|b\rangle_A$ and $|c\rangle_A$ have the energy values $\hbar\omega_a$, $\hbar\omega_b$ and $\hbar\omega_c$, respectively. The transition $|a\rangle_A \longrightarrow |b\rangle_A$ ($|a\rangle_A \longrightarrow |c\rangle_A$) is effected through k (s) photons of the field mode $\hat{a}$ with an eigenfrequency Ω. The detunings of the levels $|a\rangle_A, |b\rangle_A$ and $|a\rangle_A, |c\rangle_A$ are $\Delta_1 = \omega_{ab} - k\Omega$ and $\Delta_2 = \omega_{ac} - s\Omega$, respectively.

by the presence of bosonic variables describing the quantized electromagnetic mode and of pseudo-spin atomic operators.

The organization of this paper is as follows. In section 2 we introduce our Hamiltonian model and give an exact expression for the unitary evolution operator U_t in the frame of the dressed-state formalism. In section 3 we employ the analytical results obtained in section 2 to investigate the properties of the entanglement degree due to the quasi-mutual entropy. We devote section 4 to our discussion in which we assume that the electromagnetic field is in different states such as coherent and squeezed states, and that the atom is initially in the mixed state. Finally, a summary of the main points of this work ends the paper and a few avenues for further investigations are indicated.

2. Theoretical model

During the last decade many theoretical and experimental efforts have been made in order to study two-photon processes involving atoms inside a cavity, stimulated by the experimental realization of a two-photon micromaser. The two-photon process is also an efficient way of generating nonclassical states of the electromagnetic field. In this paper, we consider the atomic system displayed in figure 1. We study a three-level atom injected into a cavity field in a Λ-configuration, where the dipole-allowed transitions between the upper level $|a\rangle_A$ and the lower levels $|b\rangle_A$ and $|c\rangle_A$ are non-resonant with the cavity mode. The transition between the two lower levels is dipole forbidden. Furthermore, we assume the interaction including an arbitrary form of nonlinearity of the intensity-dependent coupling. In the rotating wave approximation, the interaction of the cavity mode with the injected atom is described by the Hamiltonian

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{\text{in}}. \tag{1}$$

Here $\hat{H}_A$ and $\hat{H}_F$ describe the free atom and free field, respectively, and $\hat{H}_{\text{in}}$ describes the atom–field interaction in rotating wave approximations, where

$$\begin{aligned}
\hat{H}_A &= \omega_a|a\rangle_A\langle a|_A + \omega_b|b\rangle_A\langle b|_A + \omega_c|c\rangle_A\langle c|_A \\
\hat{H}_F &= \Omega\hat{a}^\dagger\hat{a} \\
H_{\text{in}} &= \gamma_1\{f(\hat{a}^\dagger\hat{a})\otimes\hat{a}^k\otimes|a\rangle_A\langle b|_A + \hat{a}^{\dagger k}\otimes|b\rangle_A\langle a|_A\otimes f(\hat{a}^\dagger\hat{a})\} \\
&\quad + \gamma_2\{f(\hat{a}^\dagger\hat{a})\otimes\hat{a}^s\otimes|a\rangle_A\langle c|_A + \hat{a}^{\dagger s}\otimes|c\rangle_A\langle a|_A\otimes f(\hat{a}^\dagger\hat{a})\}.
\end{aligned} \tag{2}$$

The operator $|i\rangle_A\langle i|_A$ $(i = a, b, c)$ describes the atomic population of level $|i\rangle_A$ with energy ω_i and the operator $|i\rangle_A\langle j|_A$ $(i \neq j)$ describes the transition from level $|i\rangle_A$ to level $|j\rangle_A$. The transition between the levels $|a\rangle_A$ and $|b\rangle_A$ ($|a\rangle_A$ and $|c\rangle_A$) is effected through k (s) photons with frequency ω_{ab} (ω_{ac}), where the photon numbers k and s are positive integers. We denote by $\hat{a}$ and $\hat{a}^\dagger$, respectively, the annihilation and the creation operators for the mode of the cavity field, and Ω the field frequency. We denote by $\gamma_i f(a^\dagger a)$ an arbitrary intensity-dependent coupling. The parameters γ_i are corresponding atom–field coupling constants. In the standard predictive picture of quantum mechanics, the absorption and emission of multi-photons between different atomic levels are of course familiar processes. Experimental implementation of the theoretical idea proposed herein would possibly depend on the use of trapped neutral atoms or ions in a high-Q cavity, or an atomic beam in transit through the cavity [10].

Mixing two entangled pure states could result in a mixed state with entanglement much less than the average entanglement of the states mixed. Now, we suppose that the initial state of the atom is given by

$$\rho_A(0) = \varsigma_1|a\rangle_A\langle a|_A + \varsigma_2|b\rangle_A\langle b|_A + \varsigma_3|c\rangle_A\langle c|_A \in \mathfrak{S}_A \tag{3}$$

where $\varsigma_i \geqslant 0$, and $\varsigma_1 + \varsigma_2 + \varsigma_3 = 1$. Also we suppose that the initial state of the field is given by

$$\rho_F(0) = |\varpi\rangle\langle\varpi| \in \mathfrak{S}_F \tag{4}$$

where $|\varpi\rangle = \sum_{n=0}^{\infty} b_n|n\rangle$ and $b_n^2 = |\langle\varpi|n\rangle|^2$ being the probability distribution of photon number for the initial state, with the normalization condition $\sum_{n=0}^{\infty} b_n^2 = 1$. The continuous map $\mathcal{E}_t^*$ describing the time evolution between the atom and the field is defined by the unitary evolution operator generated by $\hat{H}$ such that

$$\begin{aligned} &\mathcal{E}_t^* : \mathfrak{S}_A \longrightarrow \mathfrak{S}_A \otimes \mathfrak{S}_F \\ &\mathcal{E}_t^*\rho = \hat{U}_t(\rho_A(0) \otimes \rho_F(0))\hat{U}_t^* \\ &\hat{U}_t \equiv \exp(-\mathrm{i}t\hat{H}). \end{aligned} \tag{5}$$

This unitary evolution operator U_t can be written as

$$\hat{U}_t = \sum_{n=0}^{\infty}\sum_{j=1}^{3} \exp\left(-\mathrm{i}tE_j^{(n)}\right)\left|\Psi_j^{(n)}\right\rangle\left\langle\Psi_j^{(n)}\right| \tag{6}$$

where

$$E_j^{(n)} = (j-1)2^{(2-j)}(\delta + (-1)^j\mu(n)) \tag{7}$$

are the eigenvalues with

$$\mu(n) = \sqrt{V_n^2 + U_n^2 + \delta^2} \tag{8}$$

where $\delta = \Delta/2$ and Δ is the detuning between the atomic frequency and the cavity mode. In order to keep the mathematical effort reasonable and to avoid too lengthy expressions, we have assumed $\Delta_1 = \Delta_2 = \Delta$, where $\Delta_1 = \omega_{ab} - k\Omega$ and $\Delta_2 = \omega_{ac} - s\Omega$; $V_n = \gamma_1 f(n)\sqrt{(n+k)!/n!}$ and $U_n = \gamma_2 f(n)\sqrt{(n+s)!/n!}$. The parameter $\mu(n)$ is a modified Rabi frequency. Hence we can easily express the eigenvectors of an atom in the cavity in the interaction picture in the form

$$\left|\Psi_j^{(n)}\right\rangle = C_{j1}^{(n)}\Phi_1^{(n)} + C_{j2}^{(n)}\Phi_2^{(n)} + C_{j3}^{(n)}\Phi_3^{(n)} \tag{9}$$

where $\left(\Phi_1^{(n)}, \Phi_2^{(n)}, \Phi_3^{(n)}\right) = (|n+k\rangle \otimes |b\rangle_A, |n\rangle \otimes |a\rangle_A, |n+s\rangle \otimes |c\rangle_A)$, and $C_{ji}^{(n)}$ are given by

$$\begin{pmatrix} C_{11}^{(n)} C_{12}^{(n)} C_{13}^{(n)} \\ C_{21}^{(n)} C_{22}^{(n)} C_{23}^{(n)} \\ C_{31}^{(n)} C_{32}^{(n)} C_{33}^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{U_n}{\mu_1(n)} & 0 & \frac{-V_n}{\mu_1(n)} \\ \frac{V_n}{\sqrt{2\mu^2(n)+\Delta\mu(n)}} & \frac{\delta+\mu_n}{\sqrt{2\mu^2(n)+\Delta\mu(n)}} & \frac{U_n}{\sqrt{2\mu^2(n)+\Delta\mu(n)}} \\ \frac{V_n}{\sqrt{2\mu^2(n)-\Delta\mu(n)}} & \frac{\delta-\mu_n}{\sqrt{2\mu^2(n)-\Delta\mu(n)}} & \frac{U_n}{\sqrt{2\mu^2(n)-\Delta\mu(n)}} \end{pmatrix} \tag{10}$$

$\mu_1(n) = \sqrt{V_n^2 + U_2^2}$. Having obtained the explicit form of the unitary evolution operator U_t, the eigenvalues and the eigenfunctions for the system under consideration, we are therefore in a position to discuss the entanglement of the system.

3. Derivation of the entanglement degree

In pure-state quantum mechanics the state of the system is usually represented by a (normalized) wavefunction, which is a (unit) vector in a Hilbert space. If the system is in the pure state $|\psi(t)\rangle$ then $\rho(t)$ is simply the projector onto this state, i.e.,

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \tag{11}$$

in such a way that $\rho^2(t) = \rho(t)$ and $\mathrm{Tr}\,\rho^2(t) = 1$. A mixed state instead is defined by the class of states which satisfy the inequality $\mathrm{Tr}\,\rho^2 \leqslant 1$. On the other hand, the existence of entangled states within quantum mechanics is one of the most striking features of the theory. These states have the potential to show nontrivial nonclassical effects [9]. Considering mixed states, several entanglement measures have been defined in this case [11, 12]. The set of disentangled states $\mathfrak{D}$ is usually considered as the set of all states which can be written as convex combinations of pure tensor states:

$$\mathfrak{D} := \{\rho \in \tau | \qquad \rho \equiv \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)} \qquad \sum_i p_i = 1 \qquad \rho_i^{(k)} \in \tau(\mathcal{H}_i). \tag{12}$$

To quantify the amount of entanglement of a state σ is to define a distance of σ to the set $\mathfrak{D}$ [11], so that the entanglement $\mathcal{E}$ of σ is given by

$$\mathcal{E}(\sigma) := \min_{\rho\in\mathfrak{D}} \mathfrak{D}(\sigma\|\rho). \tag{13}$$

Here $\mathfrak{D}$ is any measure of distance between the density matrices σ and ρ, not necessarily a distance in the metrical sense. There are several possibilities to define such a distance. The relative entropy which has been proved to be one measure satisfying all the given conditions [11], is given by

$$I_\sigma(\rho\|\sigma) \equiv \mathrm{Tr}(\sigma \log\sigma - \sigma\log\rho). \tag{14}$$

Moreover, according to the triangle inequality of Araki and Lieb [14], for entangled pure states, the $I_\sigma(\rho\|\sigma)$ becomes twice of the entropy of the induced marginal state. That is, if we want to know the degree of the entangled pure states, it is sufficient to use von Neumann entropy. However, for entangled mixed states which appear in many cases, we have to use the $I_\sigma(\rho\|\sigma)$.

The results obtained in the previous section will be applied, in this section, to derive the entanglement degree for a single three-level atom interacting with a cavity field without using the diagonal approximation method adapted in [5, 6]. With a certain unitary operator, the final state after the interaction between the atom and the field is given by

$$\begin{aligned}\mathcal{E}_t^*\rho &= \hat{U}_t(\rho_A(0)\otimes\rho_F(0))\hat{U}_t^* \\ &= \varsigma_1 U_t|a;\varpi\rangle\langle\varpi;a|U_t^* + \varsigma_2 U_t|b;\varpi\rangle\langle\varpi;b|U_t^* + \varsigma_3 U_t|c;\varpi\rangle\langle\varpi;c|U_t^*.\end{aligned} \tag{15}$$

Therefore the von Neumann entropy of the total system is given by

$$S(\mathcal{E}_t^*\rho) = -\varsigma_1 \log \varsigma_1 - \varsigma_2 \log \varsigma_2 - \varsigma_3 \log \varsigma_3. \tag{16}$$

Taking the partial trace over the atomic system, we obtain

$$\begin{aligned} \rho_t^F &= \mathrm{Tr}\,_A \mathcal{E}_t^* \rho \\ &= \varsigma_1 \sum_{i=1}^{3} |\psi_i(t)\rangle\langle\psi_i(t)| + \varsigma_2 \sum_{i=4}^{6} |\psi_i(t)\rangle\langle\psi_i(t)| + \varsigma_3 \sum_{i=7}^{9} |\psi_i(t)\rangle\langle\psi_i(t)| \end{aligned} \tag{17}$$

where

$$\begin{aligned} |\psi_1(t)\rangle &= \sum_{n=0}^{\infty} b_n \exp(-\mathrm{i}\delta t) \left(\cos \mu_n t + \mathrm{i}\delta \frac{\sin(\mu_n t)}{\mu_n}\right) |n\rangle \\ |\psi_2(t)\rangle &= -\mathrm{i} \sum_{n=0}^{\infty} b_n \exp(\mathrm{i}\delta t) V_n \frac{\sin(\mu_n t)}{\mu_n} |n+k\rangle \\ |\psi_3(t)\rangle &= -\mathrm{i} \sum_{n=0}^{\infty} b_n \exp(\mathrm{i}\delta t) U_n \frac{\sin(\mu_n t)}{\mu_n} |n+s\rangle \end{aligned} \tag{18}$$

$$\begin{aligned} |\psi_4(t)\rangle &= -\mathrm{i} \sum_{n=0}^{\infty} b_n \exp(-\mathrm{i}\delta t) V_n \frac{\sin(\mu_n t)}{\mu_n} |n\rangle \\ |\psi_5(t)\rangle &= \sum_{n=0}^{\infty} b_n \left(1 - \frac{V_n^2}{\mu_n^2} + \frac{V_n^2}{\mu_n^2} \exp(\mathrm{i}\delta t) \cos(\mu_n t) + \mathrm{i}\delta V_n^2 \exp(\mathrm{i}\delta t) \frac{\sin \mu_n t}{\mu_n^3}\right) |n+k\rangle \\ |\psi_6(t)\rangle &= \sum_{n=0}^{\infty} b_n \left(\left(\cos(\mu_n t) + \mathrm{i}\delta \frac{\sin(\mu_n t)}{\mu_n}\right) \frac{V_n U_n}{\mu_n^2} \exp(\mathrm{i}\delta t) - \frac{V_n U_n}{\mu_n^2}\right) |n+s\rangle \end{aligned} \tag{19}$$

$$\begin{aligned} |\psi_7(t)\rangle &= -\mathrm{i} \sum_{n=0}^{\infty} b_n \exp(-\mathrm{i}\delta t) U_n \frac{\sin(\mu_n t)}{\mu_n} |n\rangle \\ |\psi_8(t)\rangle &= \sum_{n=0}^{\infty} b_n \left(\left(\cos(\mu_n t) + \mathrm{i}\delta \frac{\sin(\mu_n t)}{\mu_n}\right) \frac{V_n U_n}{\mu_n^2} \exp(\mathrm{i}\delta t) - \frac{V_n U_n}{\mu_n^2}\right) |n+k\rangle \\ |\psi_9(t)\rangle &= \sum_{n=0}^{\infty} b_n \left(1 - \frac{U_n^2}{\mu_n^2} + \frac{U_n^2}{\mu_n^2} \exp(\mathrm{i}\delta t) \cos(\mu_n t) + \mathrm{i}\delta U_n^2 \exp(\mathrm{i}\delta t) \frac{\sin \mu_n t}{\mu_n^3}\right) |n+s\rangle. \end{aligned} \tag{20}$$

Then the von Neumann entropy for the reduced state $S(\rho_t^F)$ is computed by

$$S(\rho_t^F) = -\sum_{i=1}^{9} \lambda_i^F(t) \log \lambda_i^F(t) \tag{21}$$

where $\{\lambda_i^F(t)\}$ are the solutions of

$$\det[\hat{\rho(t)} - \hat{\lambda(t)} \hat{N(t)}] = 0 \tag{22}$$

where $\hat{\rho(t)}$ and $\hat{N(t)}$ are 9×9 matrices having the following elements:

$$\begin{aligned} [\hat{\rho(t)}]_{ij} &\equiv \langle\psi_i(t)|\rho_t^F|\psi_j(t)\rangle \\ [\hat{N(t)}]_{ij} &\equiv \langle\psi_i(t)|\psi_j(t)\rangle \end{aligned} \qquad (i, j = 1, 2, 3, \ldots, 9) \tag{23}$$

On the other hand, the final state of the atomic system is given by taking the partial trace over the field system,

$$\rho_t^A \equiv \mathrm{Tr}_F \mathcal{E}_t^* \rho$$
$$\equiv \Im_1|a\rangle\langle a| + \Im_2|a\rangle\langle b| + \Im_3|a\rangle\langle c| + \Im_4|b\rangle\langle a| + \Im_5|b\rangle\langle b|$$
$$+ \Im_6|b\rangle\langle c| + \Im_7|c\rangle\langle a| + \Im_8|c\rangle\langle b| + \Im_9|c\rangle\langle c| \tag{24}$$

where $\Im_i$ are given by

$$\Im_i = \sum_{k=1}^{3}\left\{Y_k C_{11}^{(n)} C_{ki}^{*(m)} + Y_{k+3} C_{21}^{(n)} C_{ki}^{*(m)} + Y_{k+6} C_{31}^{(n)} C_{ki}^{*(m)}\right\} \qquad i = 1, 2, 3$$
$$\Im_j = \sum_{k=1}^{3}\left\{Y_k C_{12}^{(n)} C_{kj-3}^{*(m)} + Y_{k+3} C_{22}^{(n)} C_{kj-3}^{*(m)} + Y_{k+6} C_{32}^{(n)} C_{kj-3}^{*(m)}\right\} \qquad j = 4, 5, 6 \tag{25}$$
$$\Im_r = \sum_{k=1}^{3}\left\{Y_k C_{13}^{(n)} C_{kr-6}^{*(m)} + Y_{k+3} C_{23}^{(n)} C_{kr-6}^{*(m)} + Y_{k+6} C_{33}^{(n)} C_{kr-6}^{*(m)}\right\} \qquad r = 7, 8, 9$$

and Y_i are given by

$$Y_i = \exp\left(-\mathrm{i}t E_{1i}^{(nm)}\right)\left\{\varsigma_1 C_{12}^{*(n)} C_{i2}^{(m)} + \varsigma_2 C_{11}^{*(n)} C_{i1}^{(m)} + \varsigma_3 C_{13}^{*(n)} C_{i3}^{(m)}\right\} \qquad i = 1, 2, 3$$
$$Y_j = \exp\left(-\mathrm{i}t E_{2j}^{(nm)}\right)\left\{\varsigma_1 C_{22}^{*(n)} C_{j2}^{(m)} + \varsigma_2 C_{21}^{*(n)} C_{j1}^{(m)} + \varsigma_3 C_{23}^{*(n)} C_{j3}^{(m)}\right\} \qquad j = 4, 5, 6 \tag{26}$$
$$Y_r = \exp\left(-\mathrm{i}t E_{3r}^{(nm)}\right)\left\{\varsigma_1 C_{32}^{*(n)} C_{r2}^{(m)} + \varsigma_2 C_{31}^{*(n)} C_{r1}^{(m)} + \varsigma_3 C_{33}^{*(n)} C_{r3}^{(m)}\right\} \qquad r = 7, 8, 9$$

where $\exp\left(-\mathrm{i}t E_{ij}^{(nm)}\right) = \exp\left(-\mathrm{i}t\left(E_i^{(n)} - E_j^{(m)}\right)\right)$.

Then the von Neumann entropy for the reduced state $S\left(\rho_t^A\right)$ is computed by

$$S\left(\rho_t^A\right) = -\sum_{i=1}^{3} \lambda_i^A(t) \log \lambda_i^A(t) \tag{27}$$

where $\lambda_i^A(t)$ is given by

$$\lambda_1^A(t) = -\tfrac{1}{3} - \tfrac{2}{3}(\sqrt{1 - 3\vartheta_1}) \cos(\beta)$$
$$\lambda_2^A(t) = -\tfrac{1}{3} + \tfrac{1}{3}(\cos(\beta) + \sqrt{3}\sin(\beta))(\sqrt{1 - 3\vartheta_1}) \tag{28}$$
$$\lambda_3^A(t) = -\tfrac{1}{3} + \tfrac{1}{3}(\cos(\beta) - \sqrt{3}\sin(\beta))(\sqrt{1 - 3\vartheta_1})$$

where

$$\beta = \frac{1}{3}\cos^{-1}\left(\frac{2 - 9\vartheta_1 - 27\vartheta_2}{2(1 - 3\vartheta_1)^{3/2}}\right)$$
$$\vartheta_1 = \Im_1\Im_9 + \Im_1\Im_5 + \Im_5\Im_9 - |\Im_6|^2 - |\Im_2|^2 - |\Im_3|^2 \tag{29}$$
$$\vartheta_2 = \Im_1\Im_5\Im_9 + \Im_2\Im_7\Im_6 - \Im_1|\Im_6|^2 - \Im_9|\Im_2|^2 - \Im_5|\Im_3|^2.$$

Using the above equations, the final expression for the entanglement degree in the three-level system takes the following form:

$$I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right) \equiv \mathrm{Tr}\left\{\mathcal{E}_t^*\rho\left(\log\mathcal{E}_t^*\rho - \log\left(\rho_t^A \otimes \rho_t^F\right)\right)\right\}$$
$$= S\left(\rho_t^A\right) + S\left(\rho_t^F\right) - S(\mathcal{E}_t^*\rho)$$
$$= -\sum_{i=1}^{9} \lambda_i^F(t) \log \lambda_i^F(t) - \sum_{i=1}^{3} \lambda_i^A(t) \log \lambda_i^A(t) + \sum_{i=1}^{3} \varsigma_i \log \varsigma_i. \tag{30}$$

It is evident that equation (30) allows us to study the entanglement degree of the system and convert from pure states into mixed states, which is crucial for many applications in quantum optics, physics and computing. As one can see, it is unlikely to express the sums in the above equations in a closed form, however for reasonably large value of $\bar{n}$, direct numerical evaluations can be performed.

4. Numerical simulations

In this section, our main focus is to study the behaviour of the entanglement degree $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right)$ equation (30). We investigate the numerical dependence of the entanglement degree on parameters such as the intensity-dependent coupling and detuning. There are some important features to note here; while the dressed-state is related to the entanglement (since the dressed states are in fact maximally entangled), they are two different concepts: states with the same dressedness can have different amounts of entanglement (as measured by the von Neumann entropy of their subsystems). Meanwhile, the weight factors b_n measure the relative importance of each of these components. In physical terms, their squares b_n^2 correspond to the probability distribution for measurements of the total excitation number operator on the initial state. In the particular case of a field initially in a coherent state, the probability distribution is given by

$$b_n^2 = \exp(\bar{n})\frac{\bar{n}^n}{n!} \tag{31}$$

where $\bar{n}$ is the mean photon number of photons present initially in the field. Let us mention that this distribution has its maximum around $\bar{n}$. The analytical expression for U_t can be obtained and the dynamical behaviour of the atom and the field variables of the system can be studied. We will first compare the entanglement degree obtained here with that obtained when the atom was initially prepared in a pure state [15] in order to show the validity of our measure. We specifically present the results for the evolution of the entanglement degree and atomic occupation probability.

It should be noted that at a special choice of the parameters ς_i such as $\varsigma_1 = 1$ ($\varsigma_2 = 1$), i.e., the atom initially in the upper (lower) state, the final state of the system becomes the pure entangled state. Therefore it is sufficient to use von Neumann entropy in order to measure the degree of entanglement for the above cases. Then the entanglement degree takes just twice the reduced von Neumann entropy i.e., $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right) = 2S\left(\rho_t^A\right)$. These situations have been considered and the reduced von Neumann entropy has been applied to analyse the quantum fluctuations [15]. In a general case (i.e., $\varsigma_1 \neq 0$ or 1), the final state does not necessarily become a pure state, so that we need to adopt the $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right)$ in order to measure the degree of entanglement in the present model. Thus our initial setting enables us to discuss the variation of the entanglement degree for different values of the parameter ς_1 of the initial atomic system.

In figure 2, we plot the entanglement degree $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right)$, the horizontal axis indicates dimensionless time $\gamma_1 t$, and for $\varsigma_1 = 0.99$. We assume the exact resonance case $\Delta = 0$, the mean photon number $\bar{n} = 5$, $\gamma_1 = \gamma_2$ and the nonlinear intensity-dependent coupling $f(n) = 1$. It is remarkable that the first maximum of the entanglement degree at $\gamma_1 t > 0$ is achieved at the collapse time, and at one-half of the revival time [13], the entanglement degree reaches its local minimum. Meanwhile, the general feature of the entanglement degree in the case ς_1 takes values such that $\varsigma_1 \approx 1$ is also almost identical to that in the previous cases (see figure 2(*a*)). We find that the maximum value of the entanglement in this case is given by $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right) \approx 2.18$. When we further increase the parameter $\varsigma_1 \approx 1$ we find

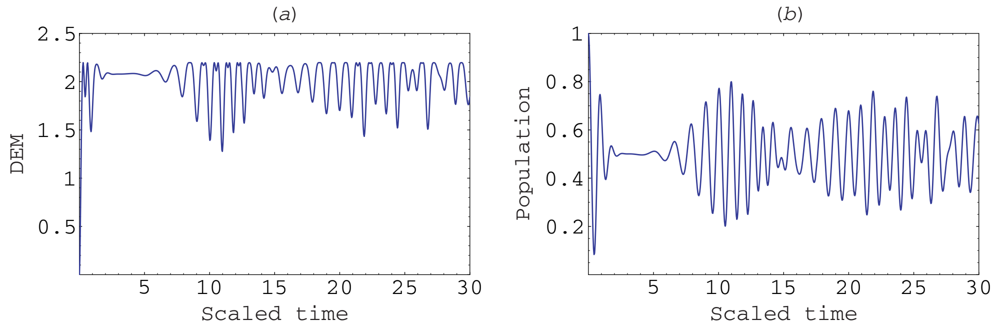

Figure 2. (*a*) The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ and (*b*) the occupation probability $P_a(t)$, as functions of the scaled time $\gamma_1 t$. The intensity-dependent atom–field coupling $f(n) = 1$, $\varsigma_1 = 0.99$, $\varsigma_3 = 0$. The intensity of the initial coherent field $\bar{n} = 5$, and the detuning parameter Δ has zero value, the number of the photons is $k = s = 1$ and $\gamma_1 = \gamma_2 = 1$.

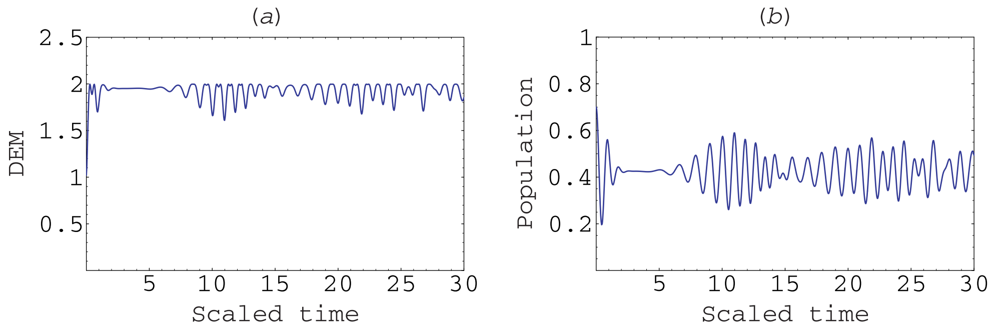

Figure 3. (*a*) The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ and (*b*) the occupation probability $P_a(t)$, as functions of the scaled time $\gamma_1 t$, $f(n) = 1$, $\varsigma_1 = 0.7$, $\varsigma_3 = 0$. The intensity of the initial coherent field $\bar{n} = 5$, and $\Delta = 0$, the number of the photons is $k = s = 1$ and $\gamma_1 = \gamma_2$.

that our degree of entanglement takes just twice the value of the von Neumann entropy i.e., $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right) \approx 2\log 3$. It has been shown that the atomic occupation probability undergoes a collapse followed by a series of revivals (see figure 2(*b*)). The collapse is due to the destructive interference of quantum Rabi flopping at different frequencies; a similar phenomenon may also occur with a classical field, however, the revivals are a purely quantum mechanical effect that originates in the discreteness of the quantum field. Another significant feature of the effect of initial state specification and entanglement is revealed in figure 3, in which we consider $\varsigma_1 = 0.7$. We see that the maximum value of the entanglement degree decreases (see figure 3(*a*)). Also, in this case we show that the amplitude of the oscillations is decreased.

To estimate the revival times of the Rabi oscillations in the limit of large one-photon detuning, we follow an analogous procedure given in [16]. We assume that the dominant contribution in the summation is from the term for which $n \approx \bar{n}$, where $\bar{n}$ is the mean photon number for which the initial photon number distribution is maximum. To single out this dominant term we rewrite $\mu_n^2 = \mu_{\bar{n}}^2 + \left(\gamma_1^2 + \gamma_2^2\right)(n - \bar{n})$, then the time of revivals t_R of the Rabi oscillations is given by $t_R = 2\pi r\Delta\big/\left(\gamma_1^2 + \gamma_2^2\right)$. Thus in the large detuning limit, the revival times are independent of the intensity of the initial field. However, as soon as we take two different values of the coupling parameters we realize that there are close-ups of the intermediate regions between the maxima of each two consecutive revivals, see figure 4. In

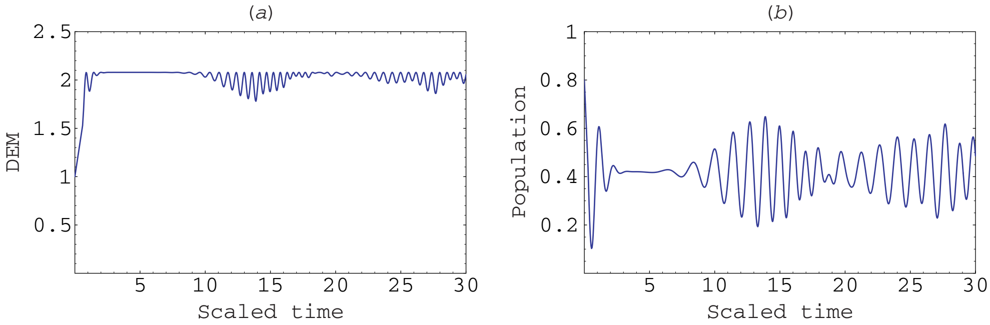

Figure 4. (*a*) The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ and (*b*) the occupation probability $P_a(t)$, as functions of the scaled time $\gamma_1 t$. The intensity-dependent atom–field coupling $f(n) = 1$, $\varsigma_1 = 0.8$, $\varsigma_3 = 0$. The intensity of the initial coherent field $\bar{n} = 5$, and the detuning parameter Δ has zero value, the number of the photons is $k = s = 1$ and $\gamma_1 = 1$, $\gamma_2 = 0.5$.

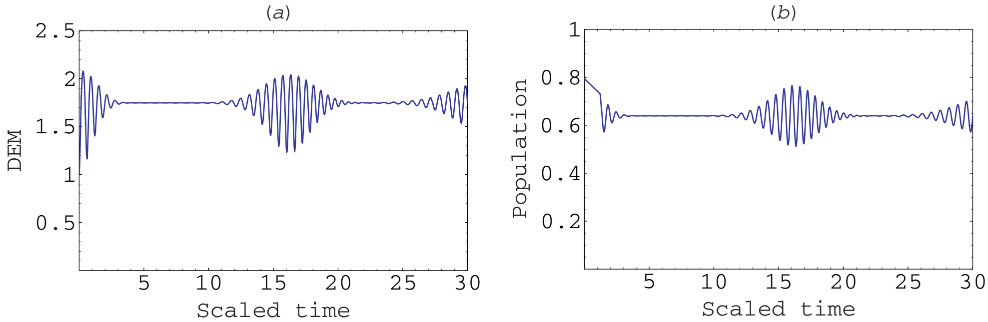

Figure 5. (*a*) The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ and (b) the occupation probability $P_a(t)$, as functions of the scaled time $\gamma_1 t$. The intensity-dependent atom–field coupling $f(n) = 1$, $\varsigma_1 = 0.8$, $\varsigma_3 = 0$. The intensity of the initial coherent field $\bar{n} = 5$, and the detuning parameter Δ has zero value, the number of the photons is $k = s = 1$ and $\gamma_1 = \gamma_2 = 1$, $\Delta/\gamma_1 = 5$.

this figure, we consider the case in which $\gamma_1 \neq \gamma_2$, such that $\gamma_1 = 1$, $\gamma_2 = 0.5$ and $\varsigma_1 = 0.8$. We see that the maximum value of the entanglement degree decreases further.

The effect of the parameter Δ which describes the mismatch between the atomic frequency and the mean frequency of the cavity mode has been considered in figure 5. We set the other parameters $\gamma_1 = \gamma_2 = 1$, $\varsigma_1 = 0.8$ and the detuning parameter $\Delta/\gamma_1 = 5$. When the detuning is considered we find that the situation has been changed. As we increase the value of the detuning we have more oscillations but with time of revivals prolonged, see for example figure 5. It is also noted that the amplitudes of the oscillation in this model are less than their counterparts for the two-level case. Finally, we point out that as we increase the value of the detuning Δ/γ_1 one can see that the revival time is also prolonged, however the period of fluctuations is decreasing. Detuning affects the revival time by elongating it and the maximum value of the entanglement degree becomes smaller and smaller. Similar to the case of a two-level atom, detuning shifted the atomic occupation probability around which it oscillates upward meaning that the energy is stored in the atomic system. It is already known [17] that different from the one-photon two-level case only the vacuum state can be a trapping state. Even this can only occur for $\Delta \gg \gamma_1, \gamma_2$, because for small detuning the system can leave the vacuum state by the emission of one photon as well as by the emission of two photons. Interaction times

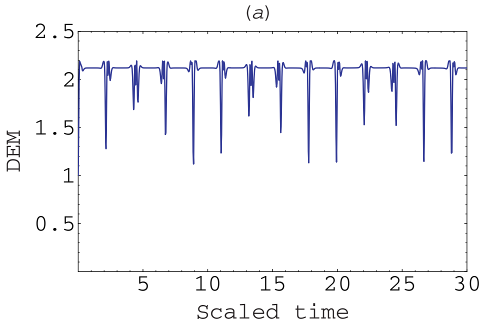

Figure 6. (*a*) The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ and (*b*) the occupation probability $P_a(t)$, as functions of the scaled time $\gamma_1 t$. The intensity-dependent atom–field coupling $f(n) = \sqrt{n}$, $\varsigma_1 = 0.8$, $\varsigma_3 = 0$. The intensity of the initial coherent field $\bar{n} = 5$, and the detuning parameter Δ has zero value, the number of the photons is $k = s = 1$ and $\gamma_1 = \gamma_2 = 1$.

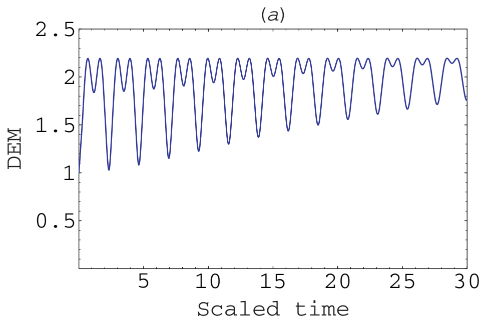

Figure 7. (*a*) The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ and (*b*) the occupation probability $P_a(t)$, as functions of the scaled time $\gamma_1 t$. The intensity-dependent atom–field coupling $f(n) = 1/\sqrt{n}$, $\varsigma_1 = 0.8$, $\varsigma_3 = 0$. The intensity of the initial coherent field $\bar{n} = 5$, and the detuning parameter Δ has zero value, the number of the photons is $k = s = 1$ and $\gamma_1 = \gamma_2 = 1$.

$t > 0$, for which the probabilities of both processes simultaneously become zero, do not exist. For $\Delta \gg \gamma_1, \gamma_2$, the occupation probability of the level $|a\rangle_A$ and, thus, the probability of a one-photon process becomes negligible.

Now we will turn our attention to the effect on the entanglement degree of the nonlinearity of the intensity-dependent coupling as an example, i.e. $f(n) = \sqrt{n}$, in figure 6, and $f(n) = 1/\sqrt{n}$ in figure 7. In particular, these forms of the intensity-dependent couplings between the atom and the cavity field remain a mere mathematical speculation but a similar form has been observed in the ion–field interaction [18]. Comparing the behaviour in figure 6, with cases considered in figure 7, we may say that the effect of the intensity coupling is rather different, where the oscillating period for $f(n) = \sqrt{n}$ is shorter than that of $f(n) = 1$ case. Also, in figure 6 there are sharp peaks observed with some kind of periodicity and more oscillations in the same period of time have been observed. This can be thought of implying that the effects on the entanglement degree of both specific intensity-dependent coupling and the initial field photon statistics can be counterbalanced in some special cases. Also, from our further calculations (which are not displayed here), we may say that the general behaviour for $f(n) = \sqrt{n}$ is similar to the two-photon case, i.e., $k = s = 2$. The case in which the intensity-dependent coupling is taken to be $f(n) = 1/\sqrt{n}$ is quite interesting where in this case the entanglement degree function oscillates around the maximum values when the time goes on. We have shown here a new phenomenon that periodic oscillations occur in the presence of intensity-dependent coupling. This difference reflects the various influences of intensity-dependent media on the interaction between atom and field. It is worth pointing out the availability of the longer period entanglement (see figure 7(*a*)). The entanglement as a physical resource is available on the condition that the entanglement should last long enough so that we can accomplish some task. For example, in order to generate the entanglement atomic state, the entanglement between the atom and the cavity field must survive long enough so that it can be transferred to the next atom via a coherent interaction. At this point, the increasingly longer period entanglement has some advantages. Although some authors use another method to prepare multiparticle entanglement, longer period entanglement is available. It should also be mentioned here that we have included the detuning parameter but we found that the entanglement degree is affected little by different values of the detuning. A slight change in ς_i therefore dramatically alters the entanglement. It should be noted that at a special choice of the nonlinear intensity-dependent coupling, the situation becomes interesting, in this case, we find that the nonlinear three-level system with an initially coherent field exhibits superstructures instead of the first-order revivals resembling those manifested by the standard three-level system.

In the following discussion we would like to highlight another special feature of the present model. It is well known that, in the case of large one-photon detuning, terms involving the ground-excited state coherence and excited state population can be adiabatically eliminated. The three-level system is then equivalent to an effective two-level system. Because of its simplicity the effective two-level system provides a good understanding of the physical phenomena responsible for the entangled state preparation and allows one to carry out analytical calculations. So, when $\delta \gg 1$, the transition of the electron can be considered as existing only between the states $|b\rangle_A$ and $|c\rangle_A$ [19]. The implementation of the entangled coherent state preparation scheme requires a single three-level Rydberg atom only, in the Λ-configuration. Previous works have concentrated essentially on the generation of entangled states of atoms [20]. Recently, several kinds of entangled states of the electromagnetic field, such as entangled coherent states [21], multimode even and odd coherent states [22], entangled photon number states [23] and so on, have been discussed in the literature. In the framework of cavity quantum electrodynamics a theoretical scheme for the generation of entangled states of photons has been proposed [24, 25]. The basic idea of most of the experimental procedures aimed at generating assigned nonclassical states of one or more modes of a single cavity field, can be traced back to the possibility of manipulating the statistical properties of the radiation field by injecting, one at a time, a flux of Rydberg atoms into the cavity [26, 27]. The high-Q cavities of life time of the order of millisecond are being used in recent experiments [26].

Squeezed states of light are nonclassical states for which the fluctuations in one of two quadrature phase amplitudes of the electromagnetic field drop below the level of fluctuations associated with the vacuum state of the field. Squeezed states therefore provide a field which is in some sense quieter than the vacuum state and hence can be employed to improve measurement precision beyond the standard quantum limits. In this case, the photon number distribution for a squeezed state, which appears in equation (4), can be written as

$$b_n^2 = \frac{J^n}{2^n n! z^{n+1}} \left| H_n\left(\frac{\beta}{\sqrt{2jz}}\right)\right|^2 \exp\left[-|\beta|^2 + \frac{J}{z}\mathrm{Re}(\beta)^2\right] \tag{32}$$

where $z = \cosh r, J = \sinh r, \beta = z\alpha + J\alpha^*, \alpha = |\alpha| \exp(\mathrm{i}\varepsilon)$ and H_n is the Hermite polynomial. We suppose here the minor axis of the ellipse, representing the direction of squeezing, is parallel to the coordinate of the field oscillator. The initial phase ε of α is

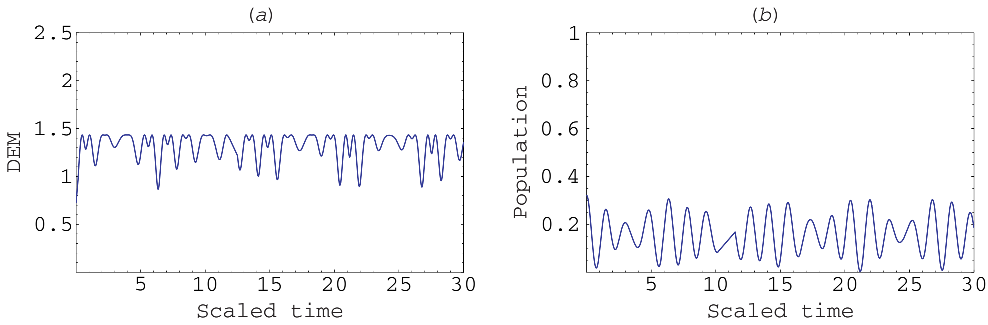

Figure 8. (*a*) The evolution of the function $I_{\mathcal{E}_t^*\rho}(\rho_t^A, \rho_t^F)$ and (*b*) the occupation probability $P_a(t)$, as functions of the scaled time $\gamma_1 t$, for an input squeezed field with $|\alpha| = \sqrt{5}, r = 0.5$. The intensity-dependent atom–field coupling $f(n) = 1, \varsigma_1 = 0.99, \varsigma_3 = 0$, and the detuning parameter Δ has zero value.

the angle between the direction of coherent excitation and the direction of squeezing. The mean photon number of this field is equal to $\bar{n} = |\alpha|^2 + \sinh^2 r$. Putting $r = 0$ we get the photon distribution for an initial coherent state with $\bar{n} = |\alpha|^2$ whereas for $\alpha = 0$ the photon distribution for an initial squeezed vacuum state with $\bar{n} = \sinh^2 r$ is recovered. The latter is oscillatory with zeros for odd n.

In figure 8, we plot the function $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right)$ which describes the entanglement degree in the case when the field is initially in a squeezed state. The initial number of photons in the cavity is $5 + \sinh^2 \frac{1}{2}$. In this case we see that the entanglement degree function oscillates around values less than the maximum values (see figure 8(*a*)). Figure 8(*b*) shows the atomic occupation probability under the same conditions as in figure 8(*a*). Intuitive pictures of the interaction between a three-level atom and an electric field commonly involve the expectation that the atomic level populations must change as both systems exchange excitations over the course of time [28, 29]. This is due to the absence of further atomic levels, which precludes the existence of destructive interference between different atomic transitions. However, in a fully quantized interaction model such as the three-level atom model, it is indeed possible to have states in which the atomic populations are completely or nearly completely trapped. This can be ultimately traced to the fact that the eigenstates of this model are entangled. From our further calculations (which are not displayed here), we may say that the dependence of the squeezed state with the intensity of the field on the entanglement degree gives behaviour similar to figure 8. While, with increasing intensity, the amplitudes of local maxima and minima are much more suppressed. Another contribution of intensity is to increase the maximum value of the entanglement degree $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right)$. Due to enhancement of intensity, the degree of entanglement fortifies. We also note that with increasing intensity of the field, nonlinear behaviour of the field is lost and its gain in coherence becomes obvious. Although the generalization of the quantum mutual entropy of entanglement to a multipartite case suggested in the present work looks rather natural, one faces a lot of difficulties while trying to compute the entanglement for a generic state. Progress can be achieved only if the state has some special properties or some symmetry.

5. Summary

Concluding this paper, we recapitulate our main results: in principle, it is possible to address the characterization of the entanglement degree due to quasi-mutual entropy. We have presented an analytical solution to a three-level system interacting with a single mode taking into account an arbitrary form of the nonlinear intensity-dependent coupling on the basis of the dressed-state formalism. It is appropriate to emphasize that the work here extends previous studies in this context [5–7]. In particular, we have explored the influence of the various parameters of the system on the entanglement degree. It is found that entanglement is affected strongly when nonlinear intensity-dependent coupling is taken into account. As expected, the maximum value of the entanglement degree decreases with decreasing occupation probability of the upper atomic level. We have found that in general the shape of revival envelopes is a direct reflection of the form of a continuous interpolation of the probability distribution. In the particular case of a field initially in a coherent state, this explains the appearance of a doublet structure in the revivals in the limit of greatest populations. It is of interest to remark that at a special choice of the nonlinear intensity-dependent coupling we have obtained longer period of the entanglement. We expect that the results of this paper can be of help for some problems, especially for quantum computation or quantum information processing, because research into the dynamical properties of multi-level atoms or trapped ions locates completely in the field of quantum computation. We also expect that this paper can lead to some other interesting discussions for systems of multi-level atoms with arbitrary form of the nonlinear intensity-dependent coupling, such as generating nonclassical states of one or more modes of a single cavity field. The next obvious step in the progression of this work would be to damp the quantum field interacting with the atom. We hope to report on such issues in a forthcoming paper.

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